INTERPOLATION AND EXTRAPOLATION

Finite difference and interpolation:

Introduction:

Many a time in practical work we come across a situation where we have to estimate a value which is not available in the given data or predict a future value. For example the census of population in India (takes place every after 10 ten years, i.e. we have census figures for 1951, 1961, 1971, etc) is as follows:

Years (x)	1931	1941	1951	1961	1981
Population (U_x)	10.4	12.9	18.1	19.1	32.5

Here the question is what is the number of population in the year (say x =) 1971 or for that matter the number of population in the year (say x =)1970. The question is what is the value of U_x for x = 1971?

Since x = 1971 (or x = 1980) for which U_x is to be estimated/calculated is an intermediate value between 1931 – 1981, the technique of estimating U_x at x = 1971 is called 'interpolation'.

On the other hand if we want to estimate the population in the year (say x =) 1992 then the technique of estimating this value is called 'extrapolation'

In the above discussion x is independent, where as U_x is dependent on x.

Interpolation consists of reading a value which lies between the two extreme points.

Extrapolation means reading a value that lies outside the two extreme points.

Definition:

Interpolation: Interpolation is an estimation of a most likely estimate in given conditions. The technique of estimating a past figure is termed as 'interpolation while that of estimating a probable figure for the future is called as 'extrapolation'

Note: *we can say that interpolation or extrapolation are only the best possible estimates under certain conditions/assumptions but not substitutes for actual value.*

Assumption of interpolation and extrapolation:

1. Generally there is uniformity between the two periods of the series.
2. Rate of change of figures from one period to another is uniform.

Methods of calculating Interpolation:

Basically there are two methods:

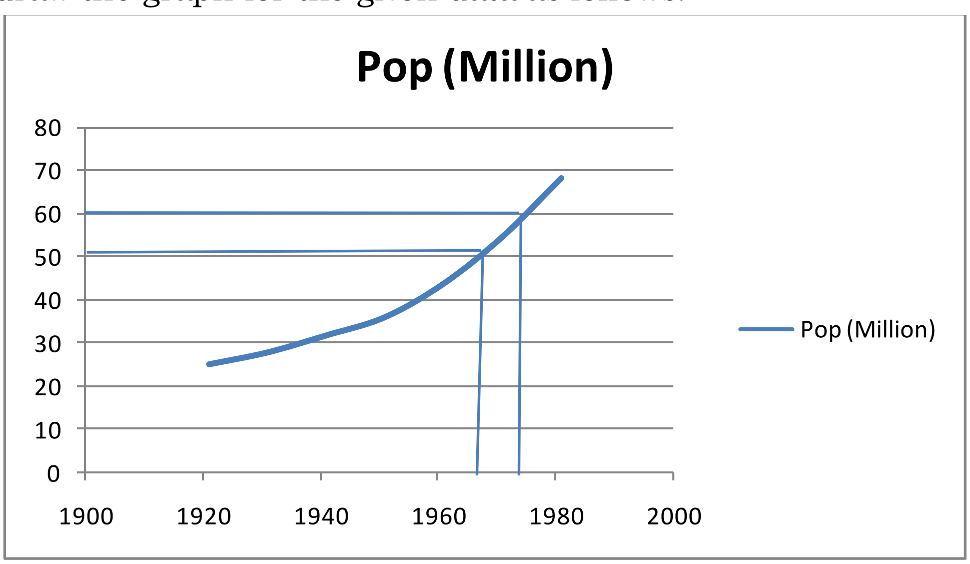
1. Graphic method.
2. Algebraic method
   a) Binomial expansion method
   b) Newton's method
   c) Langrange's method (or Divided difference method)
   d) Parabolic or Curve method.

Illustration for graphic method:

Ex. From the following data determine the population for the year 1976. Also find the number of people between 1966 and 1976.

Year	1921	1931	1941	1951	1961	1971	1981
Pop (Million)	25.13	27.91	31.87	36.11	43.91	54.79	68.38

Soln: Let us draw the graph for the given data as follows:



From the graph, number of population in 1966 is approximately 50.
Also the number of population between 1966 and 1976 is 9 mollions approximately.

Algebraic method:

Consider the following polynomial:

$$f(x) = y = U_x = 1 + 2x + 3x^2$$

We get different value of U_x corresponding to values of x as follows

X	0	1	2	3	4	5	6
U_x	1	6	17	34	57	86	121

Let us consider the series obtained by substituting the Previous Value from the Next (Example, $U_1 - U_0 = 6 - 1 = 5$ or $U_2 - U_1 = 17 - 6 = 11$ and so on)

Let us call this series as 'A'

A → 5 11 17 23 29 35

If we repeat the above operation on series 'A', we get (say series 'B') as follows

B → 6 6 6 6 6

Observe that all the terms of the series 'B' are equal to 6.

Series 'A' is called first Order Difference whereas series 'B' is called Second Order Difference and so on.

Here x's are called **'argument'** and U_x are called as **'Entry'**

Ex. Consider another polynomial (of degree 3) as follows and draw the upto 4^{th} order differences.

$$U_x = f(x) = 1 + x + x^2 + x^3 \qquad \text{for x = 0 through 6}$$

Soln: For argument values 1 through 6 the corresponding entries are calculated and the table is prepared up to 4^{th} order difference.

X	U_x	1^{st} Difference	2^{nd} Difference	3^{rd} difference	4^{th} Difference
0	1				
		3			
1	4		8		
		11		6	
2	15		14		0
		25		6	
3	40		20		0
		45		6	
4	85		26		0
		71		6	
5	156		32		
		103			
6	259				

In general,

If a polynomial is of degree 'n' then the n^{th} order difference will be constant and $(n+1)^{th}$ and higher order differences will be zero

Here 'x' denote an independent variable and argument U_x a dependent variable on x. further suppose corresponding to 'n' values of x, $x_1, x_2, \ldots x_n$ $(x_1 < x_2 < \ldots < x_n)$, then values of U_x namely $U_{x_1}, U_{x_2}, U_{x_3}, \ldots U_{x_n}$ are also known/ calculated.

Here x, x_1 , x_2 , . . . x_n are called as **"arguments"** and $U_{x_1}, U_{x_2}, U_{x_3}, \ldots U_{x_n}$ are called as **"Entries"**

Operators: Δ and $\in$

1) Difference Operator: (Δ)

Let x, x + h, x+2h, x + 3h, . . . be the arguments spaced out equally and let U_x ,U_{x+h} ,U_{x+2h} , . . . U_{x+nh} be the corresponding entries.

In other words let U_x be the function whose values are known for equidistant values of x (, given by x, x+h, x+2h, x+3h, . . . etc.)

[NOTE:
1. Here x, x + h, x+2h, x + 3h, . . ., x +nh are called arguments.
2. Here U_x ,U_{x+h} ,U_{x+2h} , . . . U_{x+nh} are called as entries.
3. For example U_x is a entry at for an argument 'x' and so on.]

We define Δ (i.e. Difference operator) as

$\Delta U_x = U_{x+h} - U_x$ for x = *x, x + h, x+2h, x + 3h, . . ., x +nh*

Now, Consider

$$\Delta U_x = U_{x+h} - U_x$$

$$\Delta U_{x+h} = U_{x+2h} - U_{x+h}$$

$$\Delta U_{x+2h} = U_{x+3h} - U_{x+2h}$$

$$\Delta U_{x+3h} = U_{x+4h} - U_{x+3h}$$

.

.

.

$$\Delta U_{x+nh} = U_{x+(n+1)h} - U_{x+nh}$$

Next,

$\Delta U_x = U_{x+h} - U_x$ …. 1st Order difference

$\Delta^2 U_x = \Delta \ \Delta U_x$ …. 2nd Order difference

$$= \Delta (U_{x+h} - U_x)$$

$$= \Delta U_{x+h} - \Delta U_x$$

$$= U_{x+2h} - U_{x+h} - (U_{x+h} - U_x)$$

$$= U_{x+2h} - 2\, U_{x+h} + U_x$$

$$\Delta^3\, U_x = \Delta\ \Delta^2\ U_x \qquad \text{.... 2nd Order difference}$$

$$= \Delta\, (U_{x+2h} - 2\, U_{x+h} + U_x)$$

$$= \Delta\, U_{x+2h} - 2\, \Delta\, U_{x+h} + \Delta U_x$$

$$= U_{x+3h} - U_{x+2h} - 2\, (U_{x+2h} - U_{x+h}) + U_{x+h} - U_x$$

$$= U_{x+3h} - 3U_{x+2h} + 3U_{x+h} - U_x$$

Similarly, we can define other higher order differences as follows,

$$\Delta^4\, U_x = \Delta\ (\Delta^3\ U_x) \qquad \text{.... 4th Oder Difference}$$

.

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$$\Delta^n\, U_x = \Delta\ (\Delta^{(n-1)}\ U_x) \qquad \text{.... nth Order Difference}$$

Table showing all these order differences is called as "**Difference Table**".

Note:

1. Difference table is as follow:

Argument	Entry	1st Difference	2nd Difference	3rd difference	4th Difference
x	U_x				
		$\Delta\, U_x$			
x+h	U_{x+h}		$\Delta^2\, U_x$		
		$\Delta\, U_{x+h}$		$\Delta^3\, U_x$	
x+2h	U_{x+2h}		$\Delta^2\, U_{x+h}$		$\Delta^4\, U_x$
		$\Delta\, U_{x+2h}$		$\Delta^3\, U_{x+h}$	
x+3h	U_{x+3h}		$\Delta^2\, U_{x+2h}$		$\Delta^4\, U_{x+h}$
		$\Delta\, U_{x+3h}$		$\Delta^3\, U_{x+2h}$	
x+4h	U_{x+4h}		$\Delta^2\, U_{x+3h}$		$\Delta^4\, U_{x+2h}$
		$\Delta\, U_{x+4h}$		$\Delta^3\, U_{x+3h}$	
x+5h	U_{x+5h}		$\Delta^2\, U_{x+4h}$		
		$\Delta\, U_{x+5h}$			
x+6h	U_{x+6h}				

2. U_a is known as the first entry in the difference table and $\Delta^1\, U_a,\ \Delta^2 U_a, \ldots$ are known as leading differences. This difference table is also known as diagonal (Forward) difference table (as the differences are shown in diagonal pattern)

E-operator:

Let x, x + h, x+2h, x + 3h, . . . be the arguments spaced out equally and let U_x ,U_{x+h} ,U_{x+2h} , . . . U_{x+nh} be the corresponding entries, where h is the length of spacing (i.e. length of interval).

E-operator is defined as,

$EU_x = U_{x+h}$ where U_x is entry corresponding to argument x.

Thus,

$$E(U_{x+h}) = U_{x+2h}$$

$$E(U_{x+2h}) = U_{x+3h}$$

$$E(U_{x+3h}) = U_{x+4h}$$

.

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$$E(U_{x+(n-1)h}) = U_{x+nh}$$

Also,

$$EU_x = U_{x+h}$$

$$E^2U_x = E(EU_x) = E(U_{x+h}) = U_{x+2h}$$

$$E^3U_x = E^2(EU_x) = E(U_{x+2h}) = U_{x+32h}$$

Relation between Δ and E.

We know that i.e. $\Delta U_x = EU_{x+h} - U_x$ and $EU_x = U_{x+h}$

Consider,

$$\Delta U_x = U_{x+h} - U_x$$

i.e. $\Delta U_x = EU_x - U_x$

i.e. $\Delta U_x = (E-1)U_x$

Thus,

$$\boxed{\Delta = (E - 1)}$$

Next, consider,

$$EU_x = U_{x+h}$$

i.e. $EU_x = \Delta U_x + U_x$

i.e. $EU_x = (\Delta + 1)U_x$

$\Rightarrow$ $E = (\Delta + 1)$

Thus,

1. $\boxed{\Delta = (E - 1)}$

2. $\boxed{E = (\Delta + 1)}$

Result:

1. If U_x is a polynomial in $'x'$ of degree $'n'$ then, $\Delta^n U_x = Constant$ and $\Delta^{n+r} U_x = 0$ for r = 1,2,3,4, ...
2. In particular, if U_xi.e. $f(x)$ is a polynomial up to degree (n-1) then $\Delta^n U_x = 0$ Also $\Delta^n U_{x+h} = 0, \Delta^n U_{x+2h} = 0$,$\Delta^n U_{x+3h} = 0$ and so on.

Ex.1. Prepare a difference table for the following data:

Years, x	1	3	5	7	9
Population, U_x	15	51	159	387	783

Soln: Let us prepare a difference table as follows:

Argument (year)	Entry (Pop)	(ΔU_x) 1st Difference	$(\Delta^2 U_x)$ 2nd Difference	$(\Delta^3 U_x)$ 3rd difference	$(\Delta^4 U_x)$ 4th Difference
1	15				
		36			
3	51		72		
		108		48	
5	159		120		0
		228		48	
7	387		168		
		396			
9	783				

Note:

1. From the table. we can conclude that the given polynomial U_x is a polynomial of degree 3 (,since $\Delta^3 U_x = 48, a\ constant$).

Ex.2. Consider the function,

$$U_x = 7 + 5x + x^2.$$

Prepare a difference table for arguments value of (x=) 0, 1, 2, 3, 4, 5.

Soln: We prepare a difference table as follows.

First we find the different values of U_x corresponding to x = 0 through 5.

Argument (year)	Entry (Pop)	(ΔU_x) 1st Difference	($\Delta^2 U_x$) 2nd Difference	($\Delta^3 U_x$) 3rd difference	($\Delta^4 U_x$) 4th Difference
0	7				
		6			
1	13		2		
		8		0	
2	21		2		0
		10		0	
3	31		2		0
		12		0	
4	43		2		
		14			
5	57				

Note: Observe that the given Polynomial is a polynomial of degree 2, therefore its 3rd order difference is zero.

Exercise:

1. Prepare a difference table for the function,
   $f(x) = 2x^3 + 3x + 2$, $x = 0\ (1)\ 5.$
2. Prepare a forward difference table from the following and hence find $f(6)$.

x	0	1	2	3	4	5
$f(x)$	0	3	8	15	24	35

3. Find $f(4)$, $f(5)$, $f(6)$ if $f(0) = -3$, $f(1) = 6$, $f(2) = 8$, $f(3) = 2$, where the third order difference is given to be constant for the function to be constant for the function.
4. Find U_5 given that the third order difference is constant.

X	0	1	2	3
U_x	- 5	- 1	9	31

Answer: 1. 2. 48 3. -13,-38,-74 4. 135

Methods of Interpolation and Extrapolation:

The methods discussed below are purely based on the assumption that the given data can be fitted with a polynomial curve whose degree is less than the number of values of U_x that are known (i.e. if n values are known then the polynomial is of degree less than or equal to (n-1). The following methods will be discussed.

1. Binomial Method.

2. Newton-Gregory forward and backward difference method.

3. Lagrange's interpolation formula for unequally spaced point.

1. Binomial Method:

Suppose arguments values $x, x+h, x+2h, x+3h, \; . \; . \; . \; . x+nh$ are known (i.e total of (n+1) terms) *for equidistant value h*, for which corresponding entry values U_x , U_{x+h} , U_{x+2h} , . . . U_{x+nh} are known, except for one. Binomial method is use to find the missing value of the entry U_x (Corresponding to the argument x). Here (n) values of U_x are known so that U_x can be consider as a polynomial up to degree (n-1).

(Therefore its $(n-1)^{th}$ order difference will be constant). Therefore its $(n)^{th}$ order difference will be zero.

i.e. $\Delta^n U_x = 0$

(For binomial of degree n has max (n +1) terms)

i.e. $(E-1)^n U_x = 0$

i.e. $[E^n + (nC_1)E^{(n-1)}(-1) + (nC_2)E^{(n-2)}(-1)^2 + \; . \; . \; . \; +(nC_n)E^0 \; (-1)^n]U_x = 0$

i.e $[E^n - (nC_1)E^{(n-1)} + (nC_2)E^{(n-2)} - \; . \; . \; . \; . \; +(nC_n) \; (-1)^n]U_x = 0$

i.e $E^n U_x - (nC_1)E^{(n-1)}U_x + (nC_2)E^{(n-2)}U_x - \; . \; . \; . \; . \; +(nC_n) \; (-1)^n U_x = 0$

i.e $U_{x+nh} - (nC_1)U_{x+(n-1)h} + (nC_2)U_{x+(n-2)h} - \; . \; . \; . \; . \; + (-1)^n U_x = 0.$

Thus, we have different polynomial depending upon the number of argument known.

NOTE: To use Binomial Methods the arguments values has to spaced out equidistant and we can find the entry value corresponding to such arguments only

NOTE:

To use the above formula, we have (n-1) known terms of U_x so that $\Delta^n U_x = 0$.

1. To find one missing values, we solve the equations $\Delta^n U_x = 0$.
2. To find two missing values, we solve two equations $\Delta^n U_x = 0$ and $\Delta^n U_{x+h} = 0$.

   This can be extended on the similar lines.

3. We can simply write the formula using the Pascol triangle as follows

n = 1						1		1					
n = 2					1		2		1				
n = 3				1		3		3		1			
n = 4			1		4		6		4		1		
n = 5		1		5		10		10		5		1	
n = 6	1		6		15		20		15		6		1

Further,

U_{x+nh}, $U_{x+(n-1)h}$, $U_{x+(n-2)h}$, . . U_x be the entries (All known except one) Renamed as

y_n , $y_{(n-1)}$, $y_{(n-2)}$, ... y_0 .

We can write the formula as follows with sign pattern as +, - , +, - etc as

n	Formula	Max Known y_n terms
2	$y_2 - 2\,y_1 + y_0 = 0$	2
3	$y_3 - 3y_2 + 3\,y_1 - y_0 = 0$	3
4	$y_4 - 4y_3 + 6y_2 - 4\,y_1 + y_0 = 0$	4
5	$y_5 - 5y_5 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$	5
6	$y_6 - 6y_5 + 15y_5 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$	6

And so on.

Examples:

Ex.1. Find the missing value from the following table

x	1	2	3	4	5
U_x	2	5	7	-	32

Soln: Observe that the arguments x = 1, 2, 3, 4, 5 are equidistant spaced out with a interval of 1. Further for argument 4 the corresponding entry U_4 is unknown.

Observe that four U_x's are known, therefore we have,

$\Delta^4 U_x = 0$

i.e. $(E-1)^4 U_x = 0$

i.e. $E^4U_x - 4E^3U_x + 6E^2U_x - 4E^1U_x + E^0U_x = 0$

i.e. $U_{x+4h} - 4U_{x+3h} + 6U_{x+2h} - 4U_{x+h} + U_x = 0$

(for x= 1 and h =1, we have)

$U_5 - 4U_4 + 6U_3 - 4U_2 + U_1 = 0$

$32 - 4(U_4) + 6(7) - 4(5) + 2 = 0$

$32 - 4(U_4) + 42 - 20 + 2 = 0$

$U_4 = 14$

Alternatively, the same problem can be worked out with the help of difference table as follows:

Let us prepare a difference table as follows:

Argument (year)	Entry (Pop)	(ΔU_x) 1st Difference	$(\Delta^2 U_x)$ 2nd Difference	$(\Delta^3 U_x)$ 3rd difference	$(\Delta^4 U_x)$ 4th Difference
1	2				
		3			
2	5		-1		
		2		$U_4 - 8$	
3	7		$U_4 - 9$		$-4U_4 + 56$
		$U_4 - 7$		$-3U_4 + 38$	
4	U_4		$-2U_4 + 29$		
		$32 - U_4$			
5	32				

Since, four values of entries are known,

$\Delta^4 U_x = 0$

$\Rightarrow \quad -4U_4 + 56 = 0$

$\Rightarrow \quad U_4 = 14$

Note:

1. The entry value 14 corresponding to the argument x = 4, is in between 1 and 5, therefore it is called interpolation.
2. On the other hand if we had calculated any entry value corresponding to the argument value outside 1 and 5, it would have been an extrapolation.

Two missing values

Ex.2. The following data gives profits of firm (in lakhs Rs). interpolate the missing figures.

x	1950	1955	1960	1965	1970	1975
U_x	7	-	13	15	-	25

Soln: Observe that the arguments x = 1950, 1955, . . . are equidistant spaced out with a interval of 5. Further two entries are not known (corresponding to the arguments 1955 and 1970 i.e. U_{1955} and U_{1970} are not known).

Observe that four U_x's are known, therefore we have,

$\Delta^4\ U_x = 0$ and $\Delta^4\ U_{x+h} = 0$

Now, $\Delta^4\ U_x = 0$

i.e. $(E-1)^4\ U_x = 0$

i.e. $E^4U_x - 4E^3U_x + 6E^2U_x - 4E^1U_x + E^0U_x = 0$

i.e. $U_{x+4h} - 4U_{x+3h} + 6U_{x+2h} - 4U_{x+h} + U_x = 0$

(for x= 1950 and h =5, we have)

$U_{1970} - 4U_{1965} + 6U_{1960} - 4U_{1955} + U_{1950} = 0$

$U_{1970} - 4(15) + 6(13) - 4 * U_{1955} + (7) = 0$

$U_{1970} - 60 + 78 - 4U_{1955} + 7 = 0$

$U_{1970} - 4U_{1955} = -25$ . . . . . (i)

Also,

$\Delta^4\ U_{x+h} = 0$

$E^4U_{x+h} - 4E^3U_{x+h} + 6E^2U_{x+h} - 4E^1U_{x+h} + E^0U_{x+h} = 0$

$U_{x+5h} - 4U_{x+4h} + 6U_{x+3h} - 4U_{x+2h} + U_{x+h} = 0$

(for x= 1950 and h =5, we have)

$U_{1975} - 4U_{1970} + 6U_{1965} - 4U_{1960} + U_{1955} = 0$

$25 - 4\ (U_{1970}) + 6(15) - 4(13) + U_{1955} = 0$

$$25 - 4U_{1970} + 90 - 52 + U_{1955} = 0$$

$$4U_{1970} - U_{1955} = 63 \qquad \ldots\ldots \quad \text{(ii)}$$

Let us solve the two equation (i) and (ii)

$$U_{1970} - 4U_{1955} = -25 \qquad \ldots\ldots \quad \text{(i)}$$

$$4U_{1970} - U_{1955} = 63 \qquad \ldots\ldots \quad \text{(ii)}$$

Solving the above two equations,

$$U_{1970} = 18.46$$

$$U_{1955} = 10.86$$

Exercise: One missing value

Ex.1. From the following table, interpolate the missing production figure.

Year	1	2	3	4	5
Production (in '000 tons)	200	220	260	-	350

(Ans:)

<u>Exercise: Two missing value</u>

Ex.1. From the following table, interpolate the missing production figure.

Year	1	2	3	4	5	6	7
Production (in '000 tons)	200	220	260	-	350	-	430

(2000, Ans:306, 390)

Newton's Method:

1) Newton's (forward Interpolation) Method:

Suppose $a,\ a+h,\ a+2h, .\ .\ .\ ,\ a+(n-1)h$ be the n arguments (spaced out equidistantly) and $U_a\ ,\ U_{a+h}\ ,\ U_{a+2h}\ ,\ \ .\ .\ .\ U_{a+(n-1)h}$ be the corresponding entries.

Suppose we want to estimates U_{a+xh} , where x is any real number.

We have,

$$U_{a+xh} = E^x U_a$$

$$= (1+\Delta)^x\ U_a$$

Expanding by binomial expansion, we have,

$$U_{a+xh} = U_a + (xC_1)1^{(x-1)}\Delta\, U_a + (xC_2)1^{(x-2)}\Delta^2 U_a + .\ .\ .\ +(xC_x)1^{(x-x)}\Delta^x\ U_a$$

$$U_{a+xh} = U_a + \frac{(x)*\ \Delta\, U_a}{1\,!} + \frac{(x)(x-1)}{2\,!}\Delta^2 U_a + \frac{(x)(x-1)(x-2)}{3\,!}\Delta^3 U_a + \ .\ .\ .$$

$$.\ .\ .\ + \frac{(x)(x-1)(x-2).\ .\ .\ .\ .(x-n+2)}{(n-1)!}\Delta^{(n-1)}\ U_a$$

The formula U_{a+xh} is known as Newton's Forward interpolation, (as it uses the forward values of $\Delta^n\ U_a$ i.e. $\Delta^1\ U_a,\ \Delta^2 U_a,\ \Delta^3 U_a,\ .\ .\ .\ .\ \Delta^{(n-1)}\ U_a$ are the values as shown in the difference table below)

Difference table:

Argument	Entry	1st Difference	2nd Difference	3rd difference	4th Difference
X	U_x				
		$\Delta\ U_x$			
x+h	U_{x+h}		$\Delta^2\ U_x$		
		$\Delta\ U_{x+h}$		$\Delta^3\ U_x$	
x+2h	U_{x+2h}		$\Delta^2\ U_{x+h}$		$\Delta^4\ U_x$
		$\Delta\ U_{x+2h}$		$\Delta^3\ U_{x+h}$	
x+3h	U_{x+3h}		$\Delta^2\ U_{x+2h}$		$\Delta^4\ U_{x+h}$
		$\Delta\ U_{x+3h}$		$\Delta^3\ U_{x+2h}$	
x+4h	U_{x+4h}		$\Delta^2\ U_{x+3h}$		$\Delta^4\ U_{x+2h}$
		$\Delta\ U_{x+4h}$		$\Delta^3\ U_{x+3h}$	
x+5h	U_{x+5h}		$\Delta^2\ U_{x+4h}$		
		$\Delta\ U_{x+5h}$			
x+6h	U_{x+6h}				

Note:

1. To use the above formula, we must first prepare a difference table.
2. **Calculation of x**:

   In above formula, x can be calculated as follows

   Let $U_{a+xh} = U_{a_0}$ (Here a_0 is the argument for which we find the entry U_{a_0})

   $\Rightarrow \quad a + x\,h = a_0$

   $\Rightarrow \quad x\,h = a_0 - a$

   $\Rightarrow \quad x = \frac{a_o - a}{h}$

   Thus, $\boxed{x = \frac{a_o - a}{h}}$

3. It is necessary that x series has to be uniformly spaced out.
4. By Newton's method we can find any value of U_x, corresponding to any value of argument 'x'. (Here 'x' is any members of the series (x- series) or can be any intermediate value).
5. Recall that, if there are n arguments known then the nth order difference will be zero. i.e. $\Delta^n U_a = 0$. Thus, it automatically reduces the terms in the above expansion.
6. U_a is known as the first entry in the difference table and $\Delta^1 U_a$, $\Delta^2 U_a$, . . . are known as leading differences.

Ex.1. From the following table, find $U_{2.5}$

x	1	2	3	4		
U_x	10	16	26	40		

Soln: Let us prepare a difference table as follows.

A	U_x	$(\Delta\, U_x)$ 1st Difference	$(\Delta^2\, U_x)$ 2nd Difference	$(\Delta^3\, U_x)$ 3rd difference	$(\Delta^4\, U_x)$ 4th Difference
1	***10***				
		6			
2	16		***4***		
		10		***0***	
3	26		4		
		14			
4	40				

To find: $U_{2.5}$

Here, Let $U_{a+xh} = U_{a_0} = U_{2.5}$

$\Rightarrow \quad a + x\,h = 2.5$

$\Rightarrow \quad x = \frac{a_o - a}{h}$ (here, a = 1 , h = 1)

$\Rightarrow \quad x = 1.5$

By Newton's Forward interpolation formula, we have,

$$U_{2.5} = U_{a+xh} = U_a + \frac{(x) * \Delta U_a}{1\,!} + \frac{(x)(x-1)}{2\,!}\Delta^2 U_a + \frac{(x)(x-1)(x-2)}{3\,!}\Delta^3 U_a$$

(Restricted the formula to the third order difference, as $\Delta^3 U_a = 0$

$$U_{2.5} = 10 + \frac{(1.5) * 6}{1\,!} + \frac{(1.5)\,(1.5-1)}{2\,!} * 4 + \frac{(1.5)(1.5-1)(1.5-2)}{3\,!} * 0$$

$= 10 + 9 + 1.5 + 0$

$= 20.5$

Thus,

$U_{2.5} = 20.5$

Ex.2. Find the general polynomial U_x and hence find U_{12} from the below table.

x	5	10	15	20		
U_x	9.6	12.9	17.1	23.2		

Soln: Let us prepare a difference table as follows.

x	U_x	(ΔU_x) 1st Difference	$(\Delta^2 U_x)$ 2nd Difference	$(\Delta^3 U_x)$ 3rd difference	$(\Delta^4 U_x)$ 4th Difference
5	***9.6***				
		3.3			
10	12.9		***0.9***		
		4.2		**1**	
15	17.1		1.9		
		6.1			
20	23.2				

<u>To find</u>: General Polynomial U_x

We assume here a 3rd O.D. as constant. Thus, we fit a three degree polynomial. We have,

$$U_x = E^{\left(\frac{x}{5}-1\right)} U_5 \qquad \left(\text{since, } E(U_x) = U_{x+h} \therefore E^{\left(\frac{x}{5}-1\right)} U_5 = U_x\right)$$

$$= (1+\Delta)^{\left(\frac{x}{5}-1\right)} U_5$$

Expanding by binomial expansion, we have,

$$= \left(1 + \left(\left(\frac{x}{5}-1\right)C_1\right)1^{\left(\frac{x}{5}-1-1\right)}\Delta + \left(\left(\frac{x}{5}-1\right)C_2\right)1^{\left(\frac{x}{5}-1-2\right)}\Delta^2 + \left(\left(\frac{x}{5}-1\right)C_3\right)1^{\left(\frac{x}{5}-1-3\right)}\Delta^3\right) U_5$$

$$= \left(1 + \left(\left(\frac{x}{5}-1\right)C_1\right) * \Delta + \left(\left(\frac{x}{5}-1\right)C_2\right) * \Delta^2 + \left(\left(\frac{x}{5}-1\right)C_3\right) * \Delta^3\right) U_5$$

$$= \left(1 + \frac{\left(\frac{x}{5}-1\right)}{1!}\Delta + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\Delta^2}{2!} + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{x}{5}-3\right)\Delta^3}{3!}\right) U_5$$

$$= \left(1 + \frac{\left(\frac{x}{5}-1\right)}{1!}\Delta + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\Delta^2}{2!} + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{x}{5}-3\right)\Delta^3}{3!}\right) U_5$$

$$= U_5 + \frac{\left(\frac{x}{5}-1\right)}{1!}\Delta U_5 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)}{2!}\Delta^2 U_5 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{x}{5}-3\right)}{3!}\Delta^3 U_5$$

$$= 9.6 + \left(\frac{x}{5}-1\right) * 3.3 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)}{2!} * 0.9 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{x}{5}-3\right)}{3!} * 1$$

Thus,

$$U_x = 9.6 + \left(\frac{x}{5}-1\right) * 3.3 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)}{2!} * 0.9 + \frac{\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{x}{5}-3\right)}{3!} * 1 \quad . \quad .(*)$$

This is the general polynomial

To find: U_{12}

Put $x = 12$ in (*)

$$U_{12} = 9.6 + \left(\frac{12}{5} - 1\right) * 3.3 + \frac{\left(\frac{12}{5} - 1\right)\left(\frac{12}{5} - 2\right)}{2!} * 0.9 + \frac{\left(\frac{12}{5} - 1\right)\left(\frac{12}{5} - 2\right)\left(\frac{12}{5} - 3\right)}{3!} * 1$$

$$= 9.6 + 4.62 + 0.252 - 0.056$$

$$= 14.416$$

Exercise: N(F 1)

Interpolate the following

Ex.1. Use a suitable method to estimate the premium for policies maturing at the age of 47 years from the following table.

Age in years	45	50	55	60	65
Premium in Rs.	287	240	208	187	171

E.x.2. Given the values,

x	4	5	6	7		
U_x	48	100	294	900		

Use the table of difference to find the values of U_x at x = 5.4

Ex.3. Using Newton's forward interpolation formula, find $Sin\ 48^0$, it given that

$Sin\ 45^0 = 0.7071$, $Sin\ 50^0 = 0.7660$, $Sin\ 55^0 = 0.8192$, $Sin\ 60^0 = 0.8660$

Exercise: N(F2)

Extrapolate the following

Ex.1. Obtained the cubic polynomial which assumes the following values and then estimate the value of y when x = 4

x	0	1	2	3		
Y	1	3	9	19		

Ex.2. The population of a town in the decennial census was as given below. Estimate the population for the year 1985 by using Newton's forward Method.

Year 'x'	1890	1900	1910	1920	1930
Population 'U_x'	45	66	82	93	102

Langrange's formula:

Let $x,\ x+h,\ x+2h,\ .\ .\ .\ .\ .,\ x+nh$ be the arguments spaced out equally and let $U_x\ ,U_{x+h}\ ,U_{x+2h}\ ,\ .\ .\ .\ U_{x+nh}$ be the corresponding entries, where h is the length of spacing (i.e. length of interval).

Since, there are n values of entries; we have U_x as a polynomial of degree equal to less than (n-1). The require formula is given by Lagrange and is known as Lagrange's interpolation formula, given by.

$$U_x = \frac{(x-x_2)(x-x_3)(x-x_4)\ .\ .\ .\ .(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\ .\ .\ .\ .(x_1-x_n)}\ U_{x_1} + \frac{(x-x_1)\qquad (x-x_3)(x-x_4)\ .\ .\ .\ .(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\ .\ .\ .\ .(x_2-x_n)}\ U_{x_2} +$$

$$.\ .\ .\ .\ .\ .\ .\ \frac{(x-x_1)\ (x-x_2)(x-x_3)(x-x_4)\ .\ .\ .\ .(x-x_{(n-1)})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)(x_n-x_4)\ .\ .\ .\ .(x_n-x_{(n-1)})}\ U_{x_n}$$

The method of calculation of above is called Lagrange's method of interpolation. This is a lengthy polynomial.

Note:

1. To use above formula the x –series need not be of equal spacing. It can be use for x – series of unequal width.

Ex.1. Find the value of U_4 from the following table.

x	0	1	2	3		
U_x	4	7	12	20		

Soln: Given four values of U_x, thus U_x is a polynomial of degree 3 or less. It is given by Lagrange's formula as,

$$U_x = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}\ U_{x_1} + \frac{(x-x_1)\qquad (x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}\ U_{x_2} +$$

$$\frac{(x-x_1)(x-x_3)\qquad (x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}\ U_{x_3} + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}\ U_{x_4}$$

$$= \frac{(4-1)(4-2)(4-3)}{(0-1)(0-2)(0-3)}4 + \frac{(4-0)(4-2)(4-3)}{(1-0)(1-2)(1-3)}7 + \frac{(4-0)\,(4-1)(4-3)}{(2-0)\,(2-1)(2-3)}12 + \frac{(4-0)(4-1)(4-2)}{(3-0)(3-1)\,(3-2)}20$$

$$= \frac{(3)(2)(1)}{(-1)(-2)(-3)}4 + \frac{(4)(2)(1)}{(1)(-1)(-2)}7 + \frac{(4)(3)(1)}{(2)(1)(-1)}12 + \frac{(4)(3)(2)}{(3)(2)(1)}20$$

$$= (-1)*4 + (4)*7 + (-6)*12 + \ 4*20$$

$$= -4 + 28 - 72 + 80$$

$$= 32$$

Thus, the extrapolated value of $U_4 = 32$

Ex.2. Find the polynomial U_x, approximating the function whose values are given below using Lagrange's method. Find $U_{4.5}$ and U_8. Given that $U_3 = 8$, $U_5 = 20$, $U_7 = 42$ and $U_9 = 75$.

Soln: Given four values of U_x, thus U_x is a polynomial of degree 3 or less. It is given by Lagrange's formula as,

$$U_x = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} U_{x_1} + \frac{(x-x_1)\quad(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} U_{x_2}$$

$$+ \frac{(x-x_1)(x-x_3)\quad(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} U_{x_3} + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} U_{x_4}$$

$$= \frac{(x-5)(x-7)(x-9)}{(3-5)(3-7)(3-9)} 8 + \frac{(x-3)(x-7)(x-9)}{(5-3)(5-7)(5-9)} 20 + \frac{(x-3)(x-5)(x-9)}{(7-3)(7-5)(7-9)} 42 + \frac{(x-3)(x-5)(x-7)}{(9-3)(9-5)(9-7)} 75$$

$$= \frac{(x-5)(x-7)(x-9)}{(-2)(-4)(-6)} 8 + \frac{(x-3)(x-7)(x-9)}{(2)(-2)(-4)} 20 + \frac{(x-3)(x-5)(x-9)}{(4)(2)(-2)} 42 + \frac{(x-3)(x-5)(x-7)}{(6)(4)(2)} 75$$

$$= -\frac{(x-5)(x-7)(x-9)}{48} 8 + \frac{(x-3)(x-7)(x-9)}{16} 20 - \frac{(x-3)(x-5)(x-9)}{16} 42 + \frac{(x-3)(x-5)(x-7)}{48} 75$$

$$= -\frac{1(x^3-21x^2+143x-315)}{6} + \frac{5(x^3-19x^2+111x-189)}{4}$$

$$-\frac{21(x^3-17x^2+87x-135)}{8} + \frac{25(x^3-15x^2+71x-105)}{16}$$

$$= x^3\left(\frac{-1}{6} + \frac{5}{4} - \frac{21}{8} + \frac{25}{16}\right) + x^2\left(\frac{21}{6} - \frac{95}{4} + \frac{357}{8} - \frac{375}{16}\right)$$

$$+x\left(-\frac{143}{6} + \frac{555}{4} - \frac{1827}{8} + \frac{1775}{16}\right) + \left(\frac{315}{6} - \frac{945}{4} + \frac{2835}{8} - \frac{2625}{16}\right)$$

$$= x^3\left(\frac{-8}{48} + \frac{60}{48} - \frac{126}{48} + \frac{75}{48}\right) + x^2\left(\frac{168}{48} - \frac{1140}{48} + \frac{2142}{48} - \frac{1125}{48}\right)$$

$$+x\left(-\frac{1144}{48} + \frac{6660}{48} - \frac{10962}{48} + \frac{5325}{48}\right) + \left(\frac{2520}{48} - \frac{11340}{48} + \frac{17010}{48} - \frac{7875}{48}\right)$$

$$= \frac{1}{48}(x^3 + 45x^2 - 121x + 315)$$

Thus, the polynomial U_x is given by,

$U_x = \frac{1}{48}\,(x^3 + 945\,x^2 - 121x + 315) \qquad . \quad . \quad . \quad . \quad . \quad . \,(*)$

<u>To find</u>: $U_{4.5}$

Put x = 4.5 in (*)

$U_{4.5} = \frac{1}{48}\,((4.5)^3 + 945\,(4.5)^2 - 121(4.5) + 315) = \ 16.10$

<u>To find</u>: U_8

Put Put x = 8 in (*)

$U_8 = 57.06$

<u>Exercise</u>:

1. Given: f(1) = 2, f(2) = 4, f(3) = 8 and f(4) =16. Find f(5) using Lagrange's formula.

2. Interpolate the value of y when x = 3, by using Lagrange's interpolation formula for the following.

X	0	1	4	5		
y	8	11	68	123		

3. Determine the polynomial form for the data in the previous exercise

X	0	1	4	5		
y	8	11	68	123		

<u>References:</u>
1. Gupta S.P. , Statistical Methods, Sultan Chand and sons.
2. Gupta C.B., Fundamentals of Statistics, Himalaya Publishing House
3. Shah R. J., Statistical Methods.